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A simple example from flat-ship theory

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This paper describes the induced pressure distribution, free-surface waves, vortical flow and wave drag of an exact solution of low-aspect-ratio flat-ship theory. An energy balance is derived which relates the spray drag, the energy carried away by the far-field waves and the vortical flow to the total wave drag.

1. Introduction

Flat-ship theory is based on the approximation that a given ship's draught is small compared with its beam and length. Low-aspect-ratio flat-ship theory assumes the secondary approximation that the beam is small compared with its length and the Froude number (based on length) is large. Flat-ship theory can be thought of as the counterpart to Michell's famous work on thin ships (beam $\ll$ length and draught). While Michell's theory leads to an explicit solution and has been studied in detail, flat-ship theory and low-aspect-ratio flat-ship theory lead to a mixed boundary-value problem which is significantly more difficult to solve.

Maruo (1967) reformulated the problem in terms of an integral equation which must be solved for the unknown pressure distribution to obtain the given hull shape. His kernel is rather troublesome to work with as it is both singular and highly oscillatory. The unknown pressure is also singular. Tuck (1975) provided some preliminary numerical solutions to the low-aspect-ratio limit of Maruo's problem. However, because of the rather wild behaviour of the kernel, the relationship between the hull shape and the calculated flow quantities is not readily obvious.

The purpose of this paper is to provide some insight into the low-aspect-ratio problem by presenting a simple analytic example worked out in some detail.

2. Problem formulation

The general problem of flat-ship theory begins with the equations describing inviscid, incompressible, irrotational steady flow past a ship. These equations represent the conservation of fluid mass, tangential flow along the ship, tangential flow along the free surface and Bernoulli's equation for the conservation of energy. In non-dimensional form (where all quantities have been scaled according to the length L and speed U of the ship relative to the undisturbed flow) these equations are respectively

$$\left.\begin{aligned} &\Phi_{xx}+\Phi_{yy}+\Phi_{zz}=0, \\ &\Phi_z=\epsilon\Phi_x f_x+\epsilon\Phi_y f_y \quad \text{on } z=\epsilon f(x,y) \quad \text{for } (x,y) \text{ on the ship}, \\ &\Phi_z=\Phi_x\xi_x+\Phi_y\xi_y \quad \text{on } z=\xi(x,y) \quad \text{for } (x,y) \text{ off the ship}, \\ &P^*+\tfrac{1}{2}F^2(\Phi_x^2+\Phi_y^2+\Phi_z^2)+z=\tfrac{1}{2}F^2. \end{aligned}\right\} \tag{1}$$

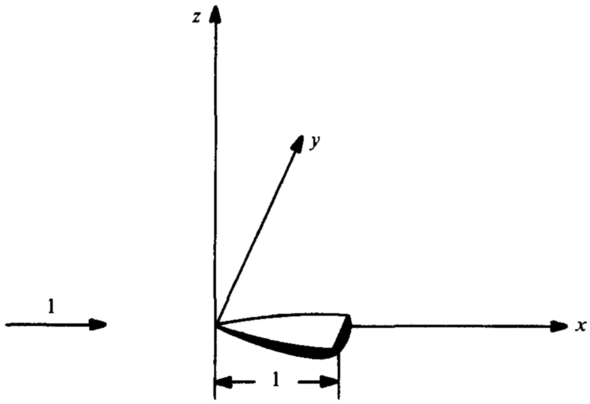

FIGURE 1. Coordinate system of flow past a ship.

In this notation, Φ is the velocity potential, $z = \epsilon f(x,y)$ describes the position of the ship's hull, ξ is the free-surface elevation and P^* is the pressure. P^* is assumed to be zero on the free surface and $P^*(x,y,\epsilon f(x,y))$ is the pressure on the ship's hull. The non-dimensional parameters are ϵ, the ratio of the ship's draught to its length, and F, the Froude number based on the length of the ship; $F = U/(gL)^{\frac{1}{2}}$ where g is the acceleration due to gravity. Figure 1 describes the coordinate system used. As this reference frame is fixed with respect to the ship, there is a uniform flow heading towards the ship from upstream infinity.

Flat-ship theory assumes that $\epsilon \ll 1$ and that (1) can be linearized using the asymptotic expansions

$$\Phi(x,y,z;\epsilon) = x + \epsilon\varphi(x,y,z) + \epsilon^2\varphi_2(x,y,z) + \dots,$$

$$\xi(x,y;\epsilon) = \epsilon\eta(x,y) + \epsilon^2\eta_2(x,y) + \dots,$$

and $$P^*(x,y,\epsilon f(x,y)) = P^*(x,y,o;\epsilon) + \dots = \epsilon P(x,y) + \dots.$$

The perturbation potential φ, surface elevation η and linearized surface pressure P, then satisfy

$$\left.\begin{aligned} &\varphi_{xx}+\varphi_{yy}+\varphi_{zz} = 0, \\ &\varphi_z = f_x(x,y) \quad \text{on } z=0 \quad \text{for } (x,y) \text{ on the ship}, \\ &\varphi_z = \eta_x(x,y) \quad \text{on } z=0 \quad \text{for } (x,y) \text{ off the ship}, \\ &F^2\varphi_x + \eta = 0 \quad \text{on } z=0 \quad \text{for } (x,y) \text{ off the ship}, \\ &P + F^2\varphi_x + f = 0 \quad \text{on } z=0 \quad \text{for } (x,y) \text{ on the ship}. \end{aligned}\right\} \tag{2}$$

Equations (2) are the general equations for flat-ship theory. While these equations are linear, they are difficult to solve since the boundary condition on the surface is prescribed in a mixed fashion.

Low-aspect-ratio flat-ship theory assumes the further simplification that the ship is slender and fast. (The ship is taken to be fast in order to preserve the character of the free waves.) The resultant problem can be thought to model the flow past a three-dimensional planing vessel. The hierarchy of approximations is draught $\ll$ beam $\ll$ length. In this limit, the changes in the x-direction are assumed to be small compared to the changes in the y- and z-directions. By defining 2δ = ship's beam/

ship's length and using the new variables $\tilde{x} = x, \tilde{y} = y/\delta$ and $\tilde{z} = z/\delta$ with $F^2 = K^2/\delta$ and $\tilde{\varphi}(\tilde{x},\tilde{y},\tilde{z}) = \varphi(x,y,z)/\delta + \ldots, \tilde{\eta}(\tilde{x},\tilde{y}) = \eta(x,y) + \ldots$, (2) yield the simplified set

$$\left.\begin{array}{rll} \tilde{\varphi}_{\tilde{y}\tilde{y}} + \tilde{\varphi}_{\tilde{z}\tilde{z}} = 0, & & \\ \tilde{\varphi}_{\tilde{z}} = f_{\tilde{x}}(\tilde{x},\tilde{y}) & \text{on } \tilde{z} = 0 & \text{for } (\tilde{x},\tilde{y}) \text{ on the ship,} \\ \tilde{\varphi}_{\tilde{z}} = \tilde{\eta}_{\tilde{x}}(\tilde{x},\tilde{y}) & \text{on } \tilde{z} = 0 & \text{for } (\tilde{x},\tilde{y}) \text{ off the ship,} \\ K^2\tilde{\varphi}_{\tilde{x}} + \tilde{\eta} = 0 & \text{on } \tilde{z} = 0 & \text{for } (\tilde{x},\tilde{y}) \text{ off the ship,} \\ P + K^2\tilde{\varphi}_{\tilde{x}} + f = 0 & \text{on } \tilde{z} = 0 & \text{for } (\tilde{x},\tilde{y}) \text{ on the ship.} \end{array}\right\} \tag{3}$$

Equations (3) still represent a mixed boundary-value problem; however, Laplace's equation must be solved only in each $(\tilde{y},\tilde{z})$ cross-plane. This allows solutions to be generated by first solving (3) for $0 \leqslant \tilde{x} \leqslant 1$ using complex-variable theory and then extending the solution behind the ship using Fourier-transform techniques. (For the rest of this paper, all reference to the low-aspect-ratio problem will be made without including the tildes.)

3. Simple example

Since φ satisfies Laplace's equation in y and z, it can be written as the real part of an analytic function G of the complex variable $\zeta = y + \mathrm{i}z$ and the parameter x for $0 \leqslant x \leqslant 1$. In order for the disturbance to decay properly at infinity, G and its derivatives must tend to zero as ζ tends to infinity (for fixed x). The surface conditions of (3) can then be rewritten in complex form as

$$K^2 G_{xx} + \mathrm{i}G_{\zeta} = -\Pi_x, \tag{4}$$

where Π is an analytic function whose real part evaluated at $z = 0$, equals the linearized surface pressure. Thus, $\mathrm{Re}(\Pi) = 0$ for $z = 0$ and (x,y) off the ship and $\mathrm{Re}(\Pi)$ is the linearized pressure on the ship's hull for $z = 0$ and (x,y) on the ship.

Since $\mathrm{Re}(\Pi_x) = 0$ for $z = 0$ and (x,y) off the ship but is not identically zero for $z = 0$ and (x,y) on the ship, (4) suggests that G can be written in the following manner. Let $G = R + \mathrm{i}I$, where R and I are analytic functions which are purely real for $z = 0$ and (x,y) off the ship but have branch points along the leading edge of the ship. R and I are also analytic inside the leading edge of the ship and both R and I have real and imaginary parts there. Since Π_x is purely imaginary and R and I are purely real for $z = 0$ and (x,y) off the ship, (4) implies R, I and Π_x satisfy the coupled equations

$$K^2 R_{xx} - I_{\zeta} = 0, \tag{5a}$$

$$K^2 \mathrm{i}I_{xx} + \mathrm{i}R_{\zeta} = -\Pi_x. \tag{5b}$$

This is a system of two equations in three unknown functions. One is, therefore, free to specify any one of these functions (as long as it has the appropriate branch points, analyticity and decay as $|\zeta| \to \infty$ as described above) and solve for the other two functions. (The resultant system of solutions must also decay appropriately as $|\zeta| \to \infty$.)

The example presented in this paper was obtained for parabolic planform ($y^2 = x$ for $0 \leqslant x \leqslant 1$) by picking $\mathrm{i}I$ to be the single term

$$\mathrm{i}I = \mathrm{i}[\zeta - (\zeta^2 - x)^{\frac{1}{2}}],$$

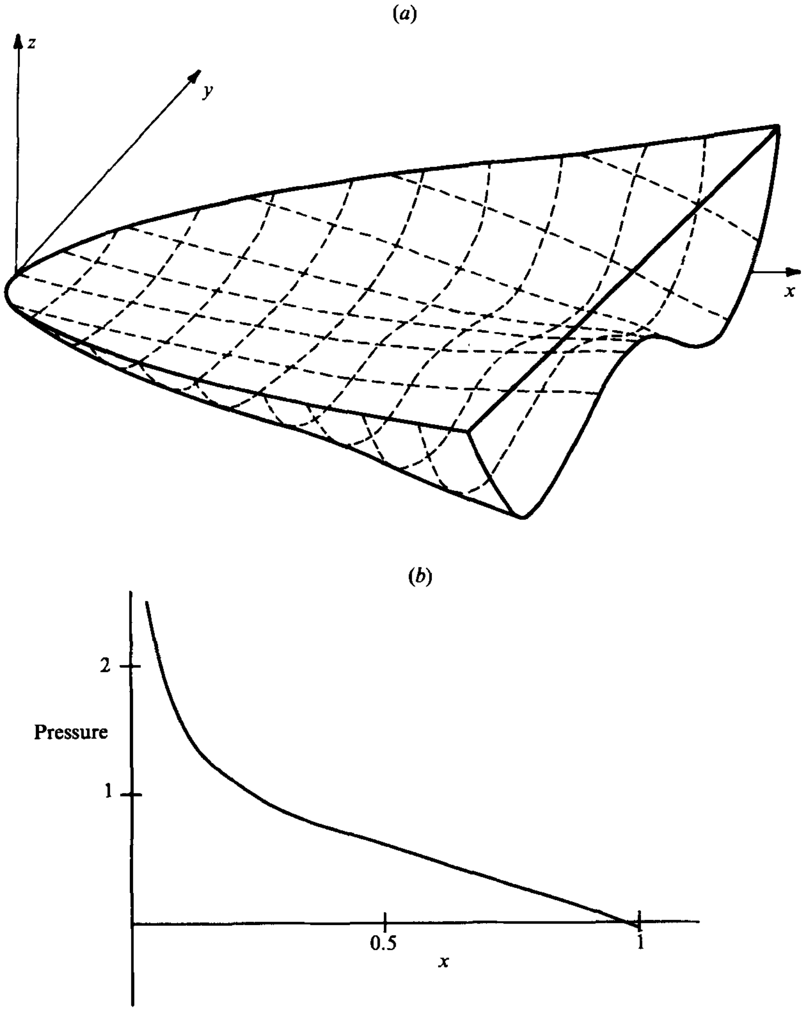

FIGURE 2. (a) Sketch of hull for $K = 1$. (b) Sketch of centreline pressure distribution for $K = 1$.

which corresponds to the flow past a flat plate at infinite speed, and solving first for R from (5a) and then for Π_x from (5b). The free functions of ζ found by integration of (5a) must be chosen such that R and R_x tend to zero as $|\zeta| \to \infty$. This potential/pressure gradient relation then defines a hull slope through the relation $f_x = \varphi_z$ for $z = 0$ and (x, y) on the ship. The linearized pressure on the ship is found by integration

$$P(x,y) = \mathrm{Re}\int \Pi_x|_{z=0}\,\mathrm{d}x + g(y),$$

where the free function $g(y)$ is chosen such that

$$P + K^2\varphi_x + f = 0 \quad \text{for } z = 0 \quad \text{and } (x,y) \text{ on the ship.}$$

In this example, $g(y)$ was chosen so that $f = 0$ along the leading edge of the ship.

This solution corresponds to a ship with hull shape

$$f = -(x-y^2) - \frac{8}{3K^2}\left[y^2(x-y^2)^{\frac{3}{2}} - \tfrac{1}{5}(x-y^2)^{\frac{5}{2}}\right]. \tag{6a}$$

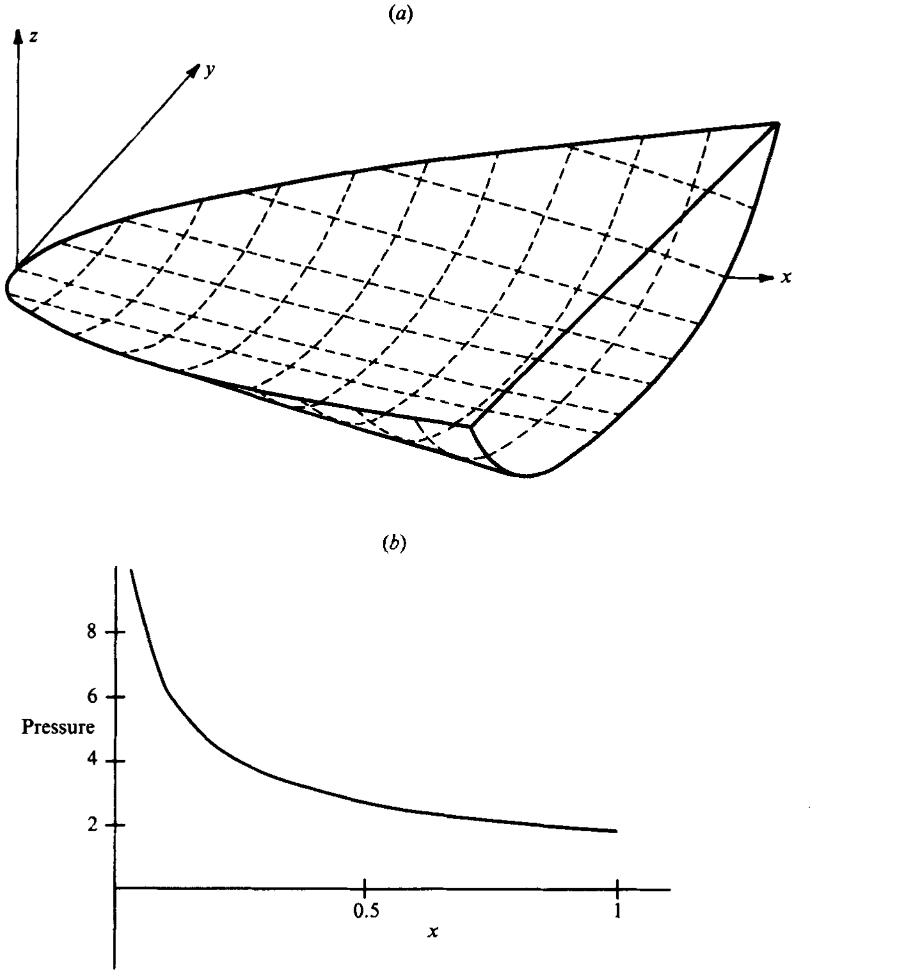

FIGURE 3. (a) Sketch of hull for $K = 2$. (b) Sketch of centreline pressure distribution for $K = 2$.

(Note that this hull shape has the mildly undesirable property that it is dependent on the parameter K.) The resultant linearized pressure on the ship is

$$P(x,y) = \frac{K^2}{2(x-y^2)^{\frac{1}{2}}} + \frac{8}{3K^2}[y^2(x-y^2)^{\frac{3}{2}} - \tfrac{1}{5}(x-y^2)^{\frac{5}{2}}] + y^2. \tag{6b}$$

The velocity potential for $0 \leqslant x \leqslant 1$ is

$$\varphi = \mathrm{Re}\left\{\mathrm{i}[\zeta-(\zeta^2-x)^{\frac{1}{2}}] - \frac{2}{3K^2}\zeta[\zeta-(\zeta^2-x)^{\frac{1}{2}}]^3 + \frac{1}{2K^2}[\zeta-(\zeta^2-x)^{\frac{1}{2}}]^4\right\}. \tag{6c}$$

A sketch of the derived hull shape and centreline pressure distribution for $K = 1$ and $K = 2$ is given in figures 2 and 3. For large values of K, this family of hulls is nearly parabolic in cross-section yet becomes significantly more wavy for small K. The centreline draught crosses zero inside the planform for $K < (\frac{8}{15})^{\frac{1}{2}}$. Then the centreline draught is negative for $0 < x < x^* = (\frac{15}{8}K^2)^{\frac{2}{3}}$ and positive for $x > x^*$. The pressure on the ship's hull has a square-root singularity along the inside of the leading edge. In this family of solutions, the singularity is positive for all values of K.

The centreline pressure is a decreasing function of x and crosses zero inside the planform for $K < (\frac{16}{15})^{\frac{1}{4}}$. Then the centreline pressure is positive for $0 < x < \hat{x} = (\frac{15}{16}K^4)^{\frac{1}{3}}$ and negative for $x > \hat{x}$. The pressure is non-singular everywhere away from the leading edge.

The vertical velocity φ_z has a square-root singularity along the outside of the ship's leading edge. The velocity and pressure singularities along the leading edge represent non-uniformities in the theory since the linearization assumptions break down there. The singularity along the outside of the ship's leading edge can be interpreted as spray.

The dynamic component of the lift is given in dimensional form by

$$\text{Dynamic lift} = \rho U^2 L^2 \epsilon\delta \iint_{\substack{\text{on}\\ \text{ship}}} -K^2 \varphi_x|_{z=0}\, \mathrm{d}x\, \mathrm{d}y = \rho U^2 L^2 \epsilon\delta \left(\frac{\pi}{2}K^2 - \frac{4}{15}\right),$$

where ρ is the density of the fluid. The static component of the lift is given in dimensional form by

$$\text{Static lift} = \rho U^2 L^2 \epsilon\delta \iint_{\substack{\text{on}\\ \text{ship}}} -f(x,y)\, \mathrm{d}x\, \mathrm{d}y = \rho U^2 L^2 \epsilon\delta \tfrac{8}{15}.$$

The total lift (in dimensional form) is the sum of these two and equals

$$\text{Total lift} = \rho U^2 L^2 \epsilon\delta \iint_{\substack{\text{on}\\ \text{ship}}} P(x,y)\, \mathrm{d}x\, \mathrm{d}y = (\rho U^2 L^2 \epsilon\delta)\left(\frac{\pi}{2}K^2 + \frac{4}{15}\right).$$

As this family of hull shapes is a function of the ship speed, it is difficult to draw any meaningful conclusions from these results. An accurate analysis of the effectiveness of the hull shape on dynamic lift (and drag) requires the solution be known for *fixed* hull shape at various high speeds. Since the two-dimensional planing problem for arbitrary hull shape was solved numerically (see Sedov 1965), it appears likely that the three-dimensional planing problem for arbitrary hull shape will also require a numerical solution. This analysis must be approached with care. (In fact, the analytic method presented in this paper was derived out of desperation when the author was unable to get a general numerical procedure to converge.) The author, still having numerical difficulties, has defined an asymptotic procedure for $K \gg 1$ and derived analytic solutions (using (5)) for the first few terms in the sequence for some simple fixed hull shapes (i.e. fixed up to the leading-order terms in the expansion) for various speeds. A lift (and drag) analysis is currently underway.

The complex variable solution, (6), valid for $0 \leqslant x \leqslant 1$, can be extended to points downstream using Fourier transforms since the pressure is known. Thus, the potential solution for all x can now be given by

$$\varphi = \left(\frac{1}{2\pi}\right)^2 \iint_{\substack{-\infty\\ \text{above }\kappa\\ \text{real axis}}}^{+\infty} \iint_{\substack{\text{on}\\ \text{ship}}} \frac{-\mathrm{i}\kappa P(\tilde{x},\tilde{y})}{K^2\kappa^2 - |l|} \mathrm{e}^{\mathrm{i}\kappa(\tilde{x}-x)}\, \mathrm{e}^{\mathrm{i}l(\tilde{y}-y)}\, \mathrm{e}^{|l|z}\, \mathrm{d}\tilde{x}\, \mathrm{d}\tilde{y}\, \mathrm{d}\kappa\, \mathrm{d}l. \tag{7}$$

The complex-variable representation for φ agrees with the Fourier-transform representation for $0 \leqslant x \leqslant 1$.

Some general statements can be made using this Fourier-transform representation of the solution. As can be expected from a high-speed theory, no disturbance exists upstream of the ship. A relatively simple expression for the far-field waves and wake can be computed asymptotically and an energy-balance relation can be derived. These are presented next.

4. Far-field waves and wake

The far-field solution can be evaluated asymptotically along rays for x and y tending to infinity from (7) using the method of stationary phase:

$$\eta \sim \frac{x^2}{4\pi^{\frac{1}{2}}K^3|y|^{\frac{5}{2}}} * \left\{\sin\left(\frac{x^2}{4K^2|y|}-\frac{\pi}{4}\right) * \iint\limits_{\substack{\text{on}\\ \text{ship}}} P(\tilde{x},\tilde{y}) \sin\left(\frac{x}{2K^2|y|}\tilde{x}\right)\cos\left(\frac{x^2}{4K^2y^2}\tilde{y}\right)\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}\right.$$

$$\left. + \cos\left(\frac{x^2}{4K^2|y|}-\frac{\pi}{4}\right) * \iint\limits_{\substack{\text{on}\\ \text{ship}}} P(\tilde{x},\tilde{y}) \cos\left(\frac{x}{2K^2|y|}\tilde{x}\right)\cos\left(\frac{x^2}{4K^2y^2}\tilde{y}\right)\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}\right\}.$$

This solution shows two families of waves whose amplitude decays like the square root of the distance to the ship and whose phase is constant along parabolas when $x^2/4K^2|y|$ is constant (see figure 4). The wave crests of these families are out of phase by 90°.

As this theory maintains the integrity of the ship's beam, it is possible to describe the flow in the wake region asymptotically for fixed y as x tends to ∞. In this example, the dominant contribution comes from the singular term in the pressure distribution:

$$\varphi \underset{y>1}{\sim} \frac{8^{\frac{1}{2}}K^2}{x^2}\left\{(y+1)^{\frac{3}{2}} \exp\left[(x^2/4K^2(y+1)^2)\,z\right] \sin\left(\frac{x^2}{4K^2(y+1)}\right)\right.$$

$$\left. + (y-1)^{\frac{3}{2}} \exp\left[(x^2/4K^2(y-1)^2)\,z\right] \cos\left(\frac{x^2}{4K^2(y-1)}\right)\right\}$$

and

$$\varphi \underset{0<y<1}{\sim} \frac{8^{\frac{1}{2}}K^2}{x^2}\left\{(y+1)^{\frac{3}{2}} \exp\left[(x^2/4K^2(y+1)^2)\,z\right] \sin\left(\frac{x^2}{4K^2(y+1)}\right)\right.$$

$$\left. + (1-y)^{\frac{3}{2}} \exp\left[(x^2/4K^2(1-y)^2)\,z\right] \sin\left(\frac{x^2}{4K^2(1-y)}\right)\right\}.$$

It can be shown that the asymptotic form for φ_x, φ_y and φ_z is equivalent to the leading-order term found by differentiation of these formulas. Since $\eta = -K^2\varphi_x$ in this region, $\eta \sim O(1/x)$ as x tends to ∞. However, the vortical flow components, φ_y and φ_z, remain of order one (in a vanishingly small region near the free surface) for all x extending to infinity. The lines $y = \pm 1$ represent caustics in this theory and must be worked out more carefully.

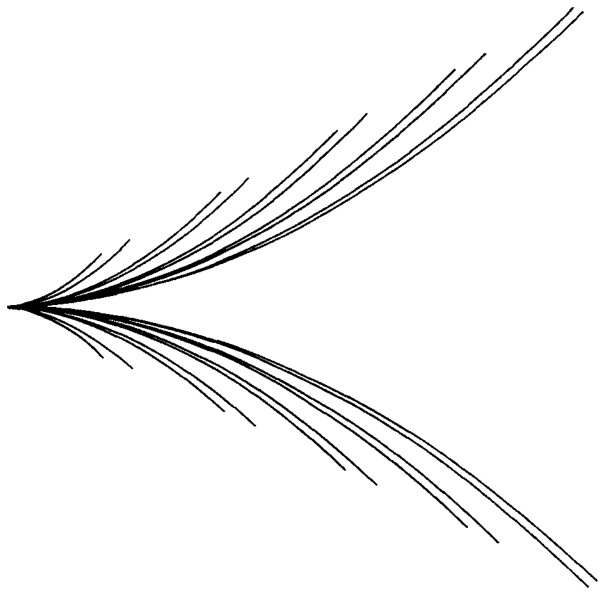

FIGURE 4. Wave crests of far-field waves.

5. Wave drag

The wave drag can be computed by integrating the pressure against the slope along the ship. In this example, the wave drag is given in dimensional form by

$$\text{Wave drag} = \rho U^2 L^2 \epsilon^2 \delta \iint_{\substack{\text{on} \\ \text{ship}}} -P\varphi_z|_{z=0}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \rho U^2 L^2 \epsilon^2 \delta \left[\frac{\pi}{2}K^2 + \frac{4}{9} + \frac{\pi}{24K^2} + \frac{2^{10}}{3\times 5\times 7\times 9\times 11\times K^4}\right].$$

A relation between the (dimensionless) wave drag, spray drag and energy carried away by the vortical flow and far-field waves can be derived as follows. Consider a control volume whose surface extends along $z = 0$ from $x = 0$ to $x = X$ ($X \to \infty$) and from $y = -Y$ to $y = +Y$ ($Y \to \infty$) except in a small region about the leading edge of the ship where the surface follows a semicircular arc C_ϵ (centred at the edge of the ship and parallel to the x-axis) as in figure 5. The bottom of the volume is located at $z = -Z$ ($Z \to \infty$) and the sides are vertical. Since φ satisfies Laplace's equation in each cross-plane,

$$\left.\begin{aligned} 0 &= \iiint -K^2\varphi_x(\varphi_{yy}+\varphi_{zz})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ \text{or}\qquad 0 &= 2\int_{x=0}^{1}\int_{C_\epsilon} -K^2\varphi_x(\nabla_T\varphi\cdot\boldsymbol{n})\,\mathrm{d}s - \iint K^2\varphi_x\varphi_y|_{y=-Y}^{Y}\,\mathrm{d}x\,\mathrm{d}z \\ &\quad -K^2\iint \varphi_x\varphi_z|_{z=-Z}^{0}\,\mathrm{d}x\,\mathrm{d}y + \iiint K^2[\varphi_{xy}\varphi_y+\varphi_{xz}\varphi_z]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \end{aligned}\right\} \tag{8}$$

$\nabla_T\varphi\cdot\boldsymbol{n}$ is the normal derivative of φ on C_ϵ. The first term in (8) is defined to be the spray drag in the limit as ϵ tends to zero and is equivalent to the momentum carried away by the flow near the leading edge of the ship. The second term does not contribute in the limit as Y tends to infinity. The third term can be expanded in the limit as $Z \to \infty$ as

$$-K^2 \iint \varphi_x \varphi_z|_{z=0}\, dx\, dy = (-\text{Wave Drag}) + \int \frac{\eta^2}{2}\bigg|_{x=X} dy.$$

The integral of $\frac{1}{2}\eta^2$ can be thought of as the energy carried away by the free-surface waves. The last term in (7) can be written as

$$\iiint K^2[\varphi_{xy}\varphi_y + \varphi_{xz}\varphi_z]\, dx\, dy\, dz = \iint K^2\left[\frac{\varphi_y^2+\varphi_z^2}{2}\right]\bigg|_{x=X} dy\, dz.$$

This integral can be interpreted as the energy carried away by the vortical flow. Thus, (8) defines

Wave Drag = Spray Drag + Energy in Waves + Energy in Vortical Flow.

It can be shown that for pressure distributions whose Fourier coefficients decay sufficiently fast, the energy in the free surface waves is asymptotically equivalent to the energy in the vortical flow as follows. Consider a surface located at $x = X$ $(X \to \infty)$ for $-\infty < y < \infty$ and $-\infty < z \leqslant 0$. Then,

$$0 = \iint K^2 \varphi \frac{\varphi_{yy}+\varphi_{zz}}{2}\bigg|_{x=X} dy\, dz$$

implies

$$\iint K^2 \frac{\varphi_y^2+\varphi_z^2}{2}\bigg|_{x=X} dy\, dz = \int K^2 \frac{\varphi\varphi_z}{2}\bigg|_{\substack{x=X\\z=0}} dy$$

$$= \int \frac{K^2}{2}\left[\frac{\partial}{\partial x}(\varphi\eta) - \varphi_x \eta\right]_{\substack{x=X\\z=0}} dy = \int \frac{-K^4}{4}\frac{\partial^2}{\partial x^2}\varphi^2\bigg|_{\substack{x=X\\z=0}} dy + \int \frac{\eta^2}{2}\bigg|_{x=X} dy.$$

However,

$$\int \frac{\partial^2}{\partial x^2}\varphi^2\bigg|_{\substack{x=X\\z=0}} dy = \frac{1}{\pi K^4}\int_{l=0}^{\infty} \frac{-2l}{K^2}\cos\left(\frac{2xl^{\frac{1}{2}}}{K}\right)\left\{\left[\iint_{\substack{\text{on}\\\text{ship}}} P(\tilde{x},\tilde{y})\cos\left(\frac{\tilde{x}l^{\frac{1}{2}}}{K}\right)\cos(l\tilde{y})\, d\tilde{x}\, d\tilde{y}\right]^2\right.$$

$$\left. -\left[\iint_{\substack{\text{on}\\\text{ship}}} P(\tilde{x},\tilde{y})\sin\left(\frac{\tilde{x}l^{\frac{1}{2}}}{K}\right)\cos(l\tilde{y})\, d\tilde{x}\, d\tilde{y}\right]^2\right\} dl$$

$$+\frac{1}{\pi K^4}\int_{l=0}^{\infty} \frac{-2l}{K^2}\sin\left(\frac{2xl^{\frac{1}{2}}}{K}\right)\left\{2\left[\iint_{\substack{\text{on}\\\text{ship}}} P(\tilde{x},\tilde{y})\sin\left(\frac{\tilde{x}l^{\frac{1}{2}}}{K}\right)\cos(l\tilde{y})\, d\tilde{x}\, d\tilde{y}\right]\right.$$

$$\left. * \left[\iint_{\substack{\text{on}\\\text{ship}}} P(\tilde{x},\tilde{y})\cos\left(\frac{\tilde{x}l^{\frac{1}{2}}}{K}\right)\cos(l\tilde{y})\, d\tilde{x}\, d\tilde{y}\right]\right\} dl.$$

This term tends to zero for pressure distributions whose Fourier coefficient

$$\iint P(\tilde{x},\tilde{y})\exp(i\tilde{x}l^{\frac{1}{2}}/K)\cos(l\tilde{y})\, d\tilde{x}\, d\tilde{y} \underset{l\to\infty}{\sim} O\left(\frac{1}{l^{\alpha}}\right) \quad \text{for } \alpha > \tfrac{3}{4}.$$

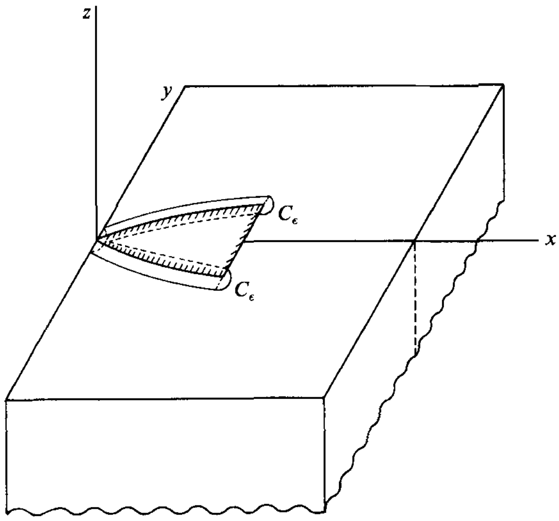

FIGURE 5. Control volume used to derive wave-drag relationship.

Then, the energy in the free-surface waves is asymptotically equivalent to the energy in the vortical flow.

In this example, the (dimensionless) spray drag is $\frac{1}{4}K^2\pi$ and the energy in the waves and in the vortex flow is

$$\frac{1}{2}\left[\frac{K^2\pi}{4}+\frac{4}{15}+\frac{\pi}{24K^2}+\frac{2^{10}}{3\times5\times7\times9\times11\times K^4}\right].$$

6. Conclusions

The flat-ship approximation is a physically realistic approximation for many ships. The theory, however, leads to a mixed boundary-value problem which requires considerable effort to solve.

Some important features of the high-speed low-aspect-ratio solution are demonstrated by the example presented in this paper. No disturbance exists ahead of the ship. There is a square-root singularity in the vertical velocity along the outside of the ship's leading edge. (This singularity represents a non-uniformity in the theory and is interpreted as spray.) The pressure also has a square-root singularity along the leading edge of the ship. Two families of far-field waves (which are out of phase by 90°) run away from the ship along parabolic trajectories. A wake region exists behind the ship which has an asymptotically small free-surface elevation but which has $O(1)$ components of vertical and sideways velocities in a vanishingly small region near the free surface all the way out to downstream infinity. The wave drag is found to be the sum of the spray drag and the energy carried away by the vortical flow and free-surface waves.

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